AN ALGORITHM FOR COMPUTING THE MULTIGRADED HILBERT DEPTH OF A MODULE

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ABSTRACT. A method for computing the multigraded Hilbert depth of a module was presented in [14]. In this paper we improve the method and we introduce an effective algorithm for performing the computations. In a particular case, the algorithm may also be easily adapted for computing the Stanley depth of the module. Thus, we completely solve an open problem proposed by Herzog in [10].

1. INTRODUCTION

In this paper we introduce an algorithm for computing the Hilbert depth of a finitely generated multigraded module M over the standard multigraded polynomial ring $R = K[X_1,\dots,X_n]$. The algorithm is based on the method presented in [14] and some extra improvements. It may also be adapted for computing the Stanley depth of M if $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$. As a consequence, we give a complete answer to the following open problem proposed by Herzog in [10]:

Problem 1. [10, Problem 1.66] *Find an algorithm to compute the Stanley depth for finitely generated multigraded R-modules M with* $\dim_K M_a \leq 1$ *for all* $a \in \mathbb{Z}^n$.

In recent years, *Stanley decompositions* of multigraded modules over R have been discussed intensively. Such decompositions, introduced by Stanley in [20], break the module M into a direct sum of *Stanley spaces*, each being of type mS where m is a homogeneous element of M, $S = K[X_{i_1},\dots,X_{i_d}]$ is a polynomial subalgebra of R and $S \bigcap \operatorname{Ann} m = 0$. One says that M has *Stanley depth* s, $\operatorname{sdepth} M = s$, if one can find a Stanley decomposition in which $d \geq s$ for each polynomial subalgebra involved, but none with s replaced by $s+1$.

The computation of the Stanley depth is not an easy task, due mainly to its combinatorial nature. A first step was done by Herzog, Vladoiu and Zheng in [13], where they introduced a method for computing the Stanley depth of a factor of a monomial ideal which was later developed into an effective algorithm by Rinaldo in [18]. Some remarkable results in the study of the Stanley depth in the multigraded case were also presented by Apel (see [1], [2]), Herzog et al. (see [11], [12]) and Popescu et al. (see [3], [17]).

Hilbert series are the most important numerical invariants of finitely generated graded and multigraded modules over R and they form the bridge from commutative algebra to its combinatorial applications (we refer here to classical results of Hilbert, Serre, Ehrhart and Stanley, see [4]). A new type of decompositions for multigraded modules M depending only on the Hilbert series of M was introduced by Bruns, Uliczka and Krattenthaler in [8] and called *Hilbert decompositions*. They are a weaker type of decompositions not requiring the summands to be submodules of M, but only vector subspaces isomorphic

to polynomial subrings. The notion of *Hilbert depth* $\operatorname{hdepth} M$ is defined accordingly. Several results concerning both the graded and multigraded cases were presented in [9], [16] and [21]. All of them are based on both combinatorial and algebraic techniques.

The contain of the paper is organized as follows. In Section 2 we recall some results concerning Hilbert depth that will be used in this paper.

Section 3 is devoted to improve the method presented in [14] by restricting as much as possible the search for a suited Hilbert decomposition. Theorem 10 shows the existence of upper–discrete Hilbert partitions of degree s for $\operatorname{hdepth} M \geq s$, so for computing the Hilbert depth it may be better to consider only this type of partitions. It is a generalization of both [18, Lemma 3.4] and [19, Lemma 3.3] (notice that, in the particular case of a factor of a monomial ideal, the Hilbert partitions coincide with the poset partitions considered by Rinaldo and Shen).

In Section 4 we introduce a *recursive* algorithm for computing the multigraded Hilbert depth of a module (see Algorithm 1). The algorithm is relative easy to implement because of its recursive form and may also be used directly for computing the Stanley depth in the case of a factor of a monomial ideal. A *non-recursive* algorithm for computing Stanley depth in the case of a factor of a monomial ideal was introduced in [18, Algorithm 1].

Hilbert decompositions are intimately related to Stanley decompositions: All Stanley decompositions are Hilbert decompositions; moreover, the latter are prerequisites to the existence of Stanley decompositions. In Section 5 we investigate how strong is this connection. We assume that $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$ and we show that Algorithm 1 may be easily modified for computing Stanley depth in this case (see Algorithm 2). This solves completely Problem 1.

We end this section with a vague remark. In the particular case of a normal affine monoid, suited Hilbert decompositions have been used with success in order to design arguable the fastest available algorithms for computing Hilbert series (see [5], [6] and [7]). It is an interesting open problem if it is possible to use suited Hilbert decompositions in order to design efficient algorithms for computing Hilbert series in other cases. This paper is likely to be of particular interest to developers of computer algebra systems.

2. PREREQUISITES

Let $R = K[X_1,\dots,X_n]$, with K a field, and let M be a finitely generated $\mathbb{Z}^n$-graded R-module. In [14] the authors presented a method for computing the multigraded Hilbert depth of M by considering Hilbert partitions of its Hilbert series. In this section we recall the method of [14].

A natural partial order on $\mathbb{Z}^n$ is defined as follows: Given $a,b \in \mathbb{Z}^n$, we say that $a \preceq b$ if and only if $a_i \leq b_i$ for $i = 1,\dots,n$. Note that $\mathbb{Z}^n$ with this partial order is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ being the componentwise minimum and maximum, respectively. We set the interval between a and b to be

$$[a,b] = \{c \in \mathbb{Z}^n \mid a \preceq c \preceq b\}.$$

We first recall a definition and a result of Ezra Miller (see [15]) which will be useful in the sequel. Let $g \in \mathbb{N}^n$. The module M is said to be $\mathbb{N}^n$-graded if $M_a = 0$ for $a \notin \mathbb{N}^n$ and M is said to be *positively g-determined* if it is $\mathbb{N}^n$-graded and the multiplication map

$\cdot X_i : M_a \longrightarrow M_{a+e_i}$ is an isomorphism whenever $a_i \geq g_i$. A characterization of positively g-determined modules is given by the following.

Proposition 2. [15, Proposition 2.5] *The module M is positively g-determined if and only if the multigraded Betti numbers of M satisfy $\beta_{0,a} = \beta_{1,a} = 0$ unless $0 \preceq a \preceq g$.*

Let

$$\bigoplus_{a\in\mathbb{Z}^n} R(-a)^{\beta_{1,a}} \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{\beta_{0,a}} \longrightarrow M \longrightarrow 0,$$

be a minimal multigraded free presentation of M and assume for simplicity, and without loss of generality, that all $\beta_{0,a} = 0$ (and *a fortiori* all $\beta_{1,a} = 0$) if $a \notin \mathbb{N}^n$.

Let $g \in \mathbb{N}^n$ be such that the multigraded Betti numbers of M satisfy the equalities $\beta_{0,a} = \beta_{1,a} = 0$ unless $0 \preceq a \preceq g$. Then, according to Proposition 2, the module M is positively g-determined. Let

$$H_M(X) = \sum_{a\in\mathbb{N}^n} H(M,a)X^a$$

be the *Hilbert series* of M and consider the polynomial

$$H_M(X)_{\preceq g} := \sum_{0\preceq a\preceq g} H(M,a)X^a.$$

For $a, b \in \mathbb{Z}^n$ such that $a \preceq b$, we set

$$Q[a,b](X) := \sum_{a\preceq c\preceq b} X^c$$

and call it the *polynomial induced by the interval* $[a,b]$.

Definition 3. We define a *Hilbert partition* of the polynomial $H_M(X)_{\preceq g}$ to be an expression

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i\in I} Q[a^i, b^i](X)$$

as a finite sum of polynomials induced by the intervals $[a^i, b^i]$.

Further we need the following notations. For $a \preceq g$ we set $Z_a = \{X_j \mid a_j = g_j\}$. Moreover we denote by $K[Z_a]$ the subalgebra generated by the subset of the indeterminates Z_a. We also define the map

$$\rho : \{0 \preceq a \preceq g\} \longrightarrow \mathbb{N}, \quad \rho(a) := |Z_a|,$$

and for $0 \preceq a \preceq b \preceq g$ we set

$$\mathscr{G}[a,b] = \{c \in [a,b] \mid c_j = a_j \text{ for all } j \in \mathbb{N} \text{ with } X_j \in Z_b\}.$$

The main result of [14] is:

Theorem 4. [14, Theorem 3.3] *The following statements hold:*

(1) *Let $\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^r Q[a^i, b^i](X)$ be a Hilbert partition of $H_M(X)_{\preceq g}$. Then*

$$\mathfrak{D}(\mathfrak{P}) : M \cong \bigoplus_{i=1}^{r} \Big(\bigoplus_{c\in\mathscr{G}[a^i,b^i]} K[Z_{b^i}](-c) \Big) \qquad [\star]$$

is a Hilbert decomposition of M. Moreover,

$$\operatorname{hdepth}\mathfrak{D}(\mathfrak{P}) = \min\{\rho(b^i) : \ i = 1,\dots,r\}.$$

(2) *Let $\mathfrak{D}$ be a Hilbert decomposition of M. Then there exists a Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$ such that*

$$\operatorname{hdepth}\mathfrak{D}(\mathfrak{P}) \geq \operatorname{hdepth}\mathfrak{D}.$$

In particular, $\operatorname{hdepth} M$ *can be computed as the maximum of the numbers* $\operatorname{hdepth}\mathfrak{D}(\mathfrak{P})$*, where $\mathfrak{P}$ runs over the finitely many Hilbert partitions of $H_M(X)_{\preceq g}$.*

We see that, in order to effectively compute the Hilbert depth of M, we may use the following corollary.

Corollary 5. [14, Corollary 3.4] *There exists a Hilbert partition*

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^{r} Q[a^i, b^i](X)$$

of $H_M(X)_{\preceq g}$ such that $\operatorname{hdepth} M = \min\{\rho(b^i) : \ i = 1,\dots,r\}$.

3. Restricting the search for a good partition

As seen in the previous section the Hilbert depth of M can be computed by considering all Hilbert partitions of $H_M(X)_{\preceq g}$. In practice, the number of possible partitions can easily become huge. For many practical purposes (for example for implementation of the method in a computer program) one needs to restrict (as much as possible) the search for a partition which will finally provide the right Hilbert depth. In this section we show that an improvement is indeed possible. Our results are extending some of the ideas presented by Giancarlo Rinaldo in [18] and Shen in [19] for computations of Stanley depth in the case of a factor of a monomial ideal to the general case of a finitely generated module.

Since many results in this section depend on a number $g \in \mathbb{N}^n$ such that M is positively g-determined, we shall assume that g is fixed and known from previous computations (for example by using Proposition 2).

Definition 6. Let B be a subset of $\mathbb{N}^n$ and $0 \leq s \leq n$. We define two subsets of B,

$$B_{<s} := \{a \in B : \rho(a) < s\} \quad \text{and} \quad B_{\geq s} := \{a \in B : \rho(a) \geq s\},$$

where ρ is the function defined in Section 2.

Our purpose is to test if M has a partition $\mathfrak{P}$ whose hdepth is equal to s. To reach this goal set $B = \{a : \ X^a$ is a monomial of the polynomial $H_M(X)_{\preceq g}\}$ and consider B as a disjoint union of the two sets defined above

$$B = B_{<s} \cup B_{\geq s}.$$

It is easy to observe that if $\mathfrak{P}$ is a Hilbert partition of $H_M(X)_{\preceq g}$, then we may write $\mathfrak{P} = A + A'$, such that

$$A = \sum_{i \in I} Q[a^i, b^i](X), \quad A' = \sum_{j \in I'} Q[a^j, b^j](X)$$

where $a^i \in B_{<s}$ and $a^j \in B_{\geq s}$ for all $i \in I$ and $j \in I'$. Then $\mathfrak{P}$ can further be refined to a new partition $\mathfrak{P}' = A + A''$ with

$$A'' = \sum_{j \in I''} Q[a^j, a^j](X)$$

where $a^j \in B_{\geq s}$ for all $j \in I''$.

Therefore, if a partition $\mathfrak{P}$ with hdepth $= s$ exists, then the part A of $\mathfrak{P}$ is composed of intervals $Q[a,b](X)$ where $a \in B_{<s}$ and $b \in B_{\geq s}$. At a first look, in order to find A, we have to consider for each element $a \in B_{<s}$ all possible candidates $b \in B_{\geq s}$ with $a \preceq b$. In the following we show that the list of candidates can be reduced considerably.

Proposition 7. *Let $P = Q[a,b](X)$ be a polynomial such that $b \preceq g$ and $\rho(a) < s \leq \rho(b)$. Then for each*

$$b^0 \in \mathrm{Min}\{x \,:\, a \preceq x \preceq b,\ \rho(x) \geq s\}$$

there exists a disjoint decomposition of P

$$P = P_0 + \sum_{i=1}^{r} P_i, \qquad (*)$$

such that P_0 is the polynomial induced by the interval $[a, b^0]$, P_i is the polynomial induced by the interval $[a^i, b^i]$, $b^r = b$ and $\rho(b^i) \geq s$ for all $i = 1, \ldots, r$.

Proof. We see that

$$\begin{aligned} P &= (X_1^{a_1} + \ldots + X_1^{b_1}) \cdots (X_n^{a_n} + \ldots + X_n^{b_n}) \\ &= X^a (1 + \ldots + X_1^{b_1 - a_1}) \cdots (1 + \ldots + X_n^{b_n - a_n}), \end{aligned}$$

so we may assume for simplicity and without loss of generality that $a = (0, \ldots, 0) \in \mathbb{N}^n$. Then we have

$$\begin{aligned} P &= (1 + X_1 + \ldots + X_1^{b_1}) \cdots (1 + X_n + \ldots + X_n^{b_n}) \\ &= P_0 + \sum_{i=1}^{r} P_i, \end{aligned}$$

where we set

$$P_0 = (1 + \ldots + X_1^{b_1^0}) \cdots (1 + \ldots + X_n^{b_n^0})$$

and

$$P_i = (1 + \ldots + X_1^{b_1^0}) \cdots (1 + \ldots + X_{i-1}^{b_{i-1}^0})(X_i^{b_i^0 + 1} + \ldots + X_i^{b_i})(1 + \ldots + X_{i+1}^{b_{i+1}}) \cdots (1 + \ldots + X_n^{b_n})$$

for all $i = 1, \ldots, r$ (in case $b_i^0 = b_i$ the term P_i is simply 0). We deduce the following description

$$b_j^i = \begin{cases} b_j^0, & \text{if } j < i, \\ b_j, & \text{otherwise.} \end{cases}$$

Since $b^0 \preceq b^i \preceq b \preceq g$, we get that $\rho(b^i) \geq \rho(b^0) \geq s$ as needed.

We claim that $(*)$ is a partition of [0,b]. To prove this it is enough to show $\mathrm{Mon}(P_i) \cap \mathrm{Mon}(P_j) \neq \emptyset$ if and only if $i = j$ and that the equality $P = P_0 + \sum_{i=1}^{r} P_i$ holds.

For the equality we will show that $\mathrm{Mon}(P) = \mathrm{Mon}(P_0) \cup \bigcup_{i=1}^{r} \mathrm{Mon}(P_i)$. We only need to show that $\mathrm{Mon}(P) \subset \mathrm{Mon}(P_0) \cup \bigcup_{i=1}^{r} \mathrm{Mon}(P_i)$ because the other equality is obvious.

Let $u \in \mathrm{Mon}(P)$, $u = X^c$. If $c_1 > b_1^0$ then $u \in \mathrm{Mon}(P_1)$, otherwise for sure $u \notin \mathrm{Mon}(P_1)$. If $c_1 \leq b_1^0$, we check if $c_2 > b_2^0$. If so $u \in \mathrm{Mon}(P_2)$, otherwise $u \notin \mathrm{Mon}(P_1) \cup \mathrm{Mon}(P_2)$. So after checking for all variables, we find that $u \in \mathrm{Mon}(P_i)$ if $c_j \leq b_j^0$, for all $j = 1, \ldots, i-1$ and $c_i > b_i^0$ or $u \in \mathrm{Mon}(P_0)$ if $c_i \leq b_i^0$ for all $i = 1, \ldots, n$. It is also clear from this description that $\mathrm{Mon}(P_i) \cap \mathrm{Mon}(P_j) \neq \emptyset$ if and only if $i = j$. □

Remark 8. In fact, in Proposition 7 we have that $\rho(b^0) = s$. Again, we may assume that $a = (0, ..., 0) \in \mathbb{N}^n$. Then, if $\rho(b^0) = t > s$, we may suppose that $b_i^0 = g_i$, for all $i = 1, \ldots, t$. We have $a < b' = (b_1^0, ..., b_s^0, 0, ..., 0) < b^0$, $\rho(b') = s$ and we get a contradiction with the minimality of b^0.

Definition 9. Let $a \in B_{<s}$. We define the set

$$B_{=s}(a) := \{x \in B_{\geq s} : a \preceq x,\ \rho(x) = s\}.$$

Theorem 10. *Assume* $\mathrm{hdepth}\, M \geq s$. *Then there exists a Hilbert partion*

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^{r} Q[a^i, b^i](X)$$

such that if $\rho(a^i) < s$ *then* $b^i \in B_{=s}(a)$.

Proof. Since $\mathrm{hdepth}\, M \geq s$, we have a partition on $H_M(X)_{\preceq g}$

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^{r} Q[a^i, b^i](X)$$

with $\rho(b^i) \geq s$. If there exists a^j such that $\rho(a^j) < s$ and b^j is not minimal we apply Proposition 7 to the polynomial induced by the interval $[a^j, b^j]$ and use the Remark 8 to end the proof. □

Example 11. Let $R = K[X_1, X_2]$ with $\deg(X_1) = (1,0)$ and $\deg(X_2) = (0,1)$. Let $M = R \oplus (X_1, X_2)R$. Then we may choose $g = (1,1)$ and

$$H_M(X_1, X_2)_{\preceq(1,1)} = 1 + 2X_1 + 2X_2 + 2X_1X_2.$$

In order to use Corollary 5 to get that $\mathrm{hdepth}\, M \geq 1$ (for details see [14, Example 3.5]), one has to compute a full Hilbert partition, for example the following

$$\mathfrak{P}_1 : (1 + X_1 + X_2 + X_1X_2) + (X_1 + X_1X_2) + X_2.$$

Since in this case $s = 1$, we have that $B_{<1} = \{(0,0)\}$ and $B_{=1}((0,0)) = \{(1,0),(0,1)\}$. By using Theorem 10 we only have to cover $(0,0)$ with an interval ending in an element of $B_{=1}((0,0))$, and the computation is reduced to obtaining one of the following two possible covers:

$$\mathfrak{C}_1 : (1+X_1), \quad \mathfrak{C}_2 : (1+X_2).$$

4. AN ALGORITHM FOR COMPUTING THE MULTIGRADED HILBERT DEPTH OF A MODULE

In this section we describe a recursive algorithm for computing the multigraded Hilbert depth of a module. The algorithm is presented in the form of a function which will be called recursively, thus realizing a backtraking search for a Hilbert partition of a given hdepth. The algorithm may also be used directly for computing Stanley depth in the case of a factor of monomial ideal. See also [18, Algorithm 1], for a non-recursive algorithm for computing Stanley depth in the case of a factor of a monomial ideal.

Algorithm 1: Function which checks if hdepth $\geq s$ recursively

Data: $g \in \mathbb{N}^n$, $s \in \mathbb{N}$ and a polynomial $P(X) = H_M(X)_{\preceq g} \in \mathbb{N}[X_1,\dots,X_n]$
Result: *true* if hdepth$M \geq s$
Boolean **CheckHilbertDepth**(g,s,P);
begin
1 **if** $P \notin \mathbb{N}[X_1,\dots,X_n]$ **then return** *false*;
2 Container $E =$ **FindElementsToCover**(g,s,P);
3 **if** size$(E) = 0$ **then return** *true*;
4 **else**
5   **for** i=begin(E) **to** i=end(E) **do**
6   Container $C[i]$:=**FindPossibleCovers**$(g,s,P,E[i])$;
7   **if** size$(C[i]) = 0$ **then return** *false*;
8   **for** j=begin$(C[i])$ **to** j=end$(C[i])$ **do**
9   Polynomial $\widetilde{P}(X) = P(X) - Q[E[i],C[i][j]](X)$;
10   **if CheckHilbertDepth**$(g,s,\widetilde{P})$ =*true* **then return** *true*;
11   **return** *false*;

At each call the function **CheckHilbertDepth** checks one interval of type $[a,b]$ to see if the polynomial induced by it may be part of a suited Hilbert partition. All possible intervals are checked in a backtraking search. A node of the searching tree is represented by a polynomial P. Below we describe the key steps.

- line 1. If the polynomial P does not have natural numbers as coefficients (positive coefficients) then it is not a sum of polynomials induced by intervals and is not a node in the searching tree.
- line 2. In this step $B_{<s}$ is computed and stored in a container. The container should provide some basic access functions (for example we want to ask for its size).
- line 3. If $B_{<s}$ is empty then we are done. We have reached a good leaf of the searching tree.

- line 4,5,8. We generate and investigate all the children of the node P.
- line 5,6. In this loop for each $a \in B_{<s}$ we compute the set $B_{=s}(a)$ (here we use Theorem 10).
- line 7. If $B_{=s}(a)$ is empty we are in a bad node and we should go back to the previous node.
- line 9,10. The child $\widetilde{P}$ is generated in line 9 and investigated in the recursive call at line 10.
- line 11. If we have reached this point then our search in this node has failed and we should go back to the previous node. If we are at the root then hdepth $< s$.

We conclude this section with a remark on the functions **FindElementsToCover** and **FindPossibleCovers**. At each node they should compute the sets $B_{<s}$ and $B_{=s}(a)$ for all $a \in B_{<s}$. For a practical implementation of the algorithm it is quite inefficient to compute them at each node. It is likely better to adjust them for the new generated child $\widetilde{P}$ and pass them down as input data for main recursive function **CheckHilbertDepth**.

5. An algorithm for computing the Stanley depth in a special case

In this section we further assume that $\dim_K M_a \le 1$ for all $a \in \mathbb{Z}^n$ and we modify Algorithm 1 for computing the Stanley depth in this case. The algorithm supplementary checks if the Hilbert partition computed by Algorithm 1 is inducing a Stanley decomposition.

Algorithm 2: Function which checks if sdepth $\ge s$ recursively

Data: $g \in \mathbb{N}^n$, $s \in \mathbb{N}$ and a polynomial $P(X) = H_M(X)_{\preceq g} \in \mathbb{N}[X_1,\dots,X_n]$
Result: *true* if sdepth $M \ge s$
Boolean **CheckStanleyDepth**(g,s,P);
begin
1 **if** $P \notin \mathbb{N}[X_1,\dots,X_n]$ **then return** *false*;
  Container $E =$ **FindElementsToCover**(g,s,P);
  if size$(E) = 0$ **then return** *true*;
  else
    for i=begin(E) **to** i=end(E) **do**
      Container $C[i]$:=**FindPossibleCovers**$(g,s,P,E[i])$;
      if size$(C[i]) = 0$ **then return** *false*;
      for j=begin$(C[i])$ **to** j=end$(C[i])$ **do**
2   **while** $a \in \mathscr{G}[E(i),C[i][j]]$ **do**
3   **if** $K[Z_{C[i][j]}] \cap \operatorname{Ann} M_a \neq 0$ **then return** *false*;
        Polynomial $\widetilde{P}(X) = P(X) - Q[E[i],C[i][j]](X)$;
        if CheckStanleyDepth$(g,s,\widetilde{P})$ =*true* **then return** *true*;
    return *false*;

The only difference from Algorithm 1 appears at lines 2,3. Here we check if the Hilbert decomposition we found is a Stanley decomposition. For this we use [14, Proposition 4.4]. The only thing to prove is that the conditions at lines 1, 3 assures that P is inducing

a Stanley decomposition. Assume that for all $a \in \mathscr{G}[E(i), C[i][j]]$ we have that $K[Z_{C[i][j]}] \cap \operatorname{Ann} M_a = 0$. Let $0 \neq m_a \in M_a$. Since $\dim_K M_a = 1$ we have that $\operatorname{Ann} m_a = \operatorname{Ann} M_a$, so $K[Z_{C[i][j]}] \cap \operatorname{Ann} m_a = 0$. Then $m_a K[Z_{C[i][j]}]$ is a Stanley space. Finally, since all the coefficients of P are ≤ 1, the condition at line 1 assures that they do not overlap.

We end with a vague remark. It is easy to see that for two intervals

$$[a_i, b_i] \cap [a_j, b_j] \neq \emptyset \Longleftrightarrow a_i \vee a_j < b_i \wedge b_j.$$

Since in this particular case the intervals do not overlap, for a practical implementation of the algorithm one may take advantage of this fact by saving the intervals and replacing the test needed at line 1.

6. Acknowledgements

The authors would like to thank Marius Vladoiu for his useful comments.

The first author was partially supported by project PN-II-RU-TE-2012-3-0161 and the second author was partially supported by project PN-II-ID-PCE-2011-3-1023, granted by the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, during the preparation of this work.

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